All Cantor sets are homeomorphic

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Entrance. A non-void topological space X is a ***generalized Cantor set*** if it is:

C1: Totally disconnected.

C2: Compact.

C3: Perfect (each point is a cluster point).

Let $\Lambda := \{\mathbf{0}, \mathbf{1}\}$ with the discrete topology. The (abstract) Cantor set is $\Lambda^{\times\mathbb{N}}$. A particular embedding of this set in $\mathbb{R}$ is as the "middle thirds" set in $[0, 1]$.

1: Theorem. *If X is a generalized Cantor set which is metrizable, then it is homeomorphic with $\Lambda^{\times\mathbb{N}}$.* ♢

First we need terminology. Here, a ***partition*** P of X is a finite collection of disjoint non-void clopen sets whose union is X. Write $\mathsf{Q} \succ \mathsf{P}$ to mean that partition Q ***refines*** P; each atom of P is a union of Q-atoms. Suppose there is a integer M such that Q partitions each atom $E \in \mathsf{P}$ into exactly M pieces. One says "Q is of degree M over P" and writes $\mathrm{Deg}(\mathsf{Q} \mid \mathsf{P}) = M$.

A seq $\vec{\mathbf{P}} = (\mathsf{P}_1, \mathsf{P}_2, \ldots)$ is a "refining sequence" if $\mathsf{P}_1 \prec \mathsf{P}_2 \prec \ldots$. For every sequence of atoms $E_1 \supset E_2 \ldots$ with $E_n \in \mathsf{P}_n$, the intersection $\bigcap_1^\infty E_n$ is non-empty, by the finite intersection property. If each such intersection is exactly one point, we say that $\vec{\mathbf{P}}$ "***separates*** points". The sequence $E_1, E_2, \ldots$ of atoms is called the ***name*** of x, where $\{x\} = \bigcap_1^\infty E_n$.

A criterion for homeomorphism

Given topological spaces X and Y, suppose there exists a separating sequence $\vec{\mathbf{P}}$ on X and another, $\vec{\mathbf{Q}}$, on Y such that

$$\forall n: \qquad \mathrm{Deg}(\mathsf{P}_{n+1} \mid \mathsf{P}_n) = \mathrm{Deg}(\mathsf{Q}_{n+1} \mid \mathsf{Q}_n).$$

Suppose further that the collection of finite intersections of $\vec{\mathbf{Q}}$-atoms forms a basis for (the topology on) Y. Then the map $f\colon X \to Y$ sending $\vec{\mathbf{P}}$-names to corresponding $\vec{\mathbf{Q}}$-names is continuous. So if X is compact then f is a cts bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

If Y is the abstract Cantor set $\Lambda^{\times\mathbb{N}}$, given positive integers $\{k_n\}_1^\infty$ there is evidently a separating sequence $\{\mathsf{Q}_n\}_1^\infty$ such that $\mathrm{Deg}(\mathsf{Q}_{n+1} \mid \mathsf{Q}_n) = 2^{k_n}$. So the theorem will follow if we can show:

2: *A generalized Cantor set X has a separating sequence $\{\mathsf{P}_n\}_1^\infty$ such that each $\mathrm{Deg}(\mathsf{P}_{n+1} \mid \mathsf{P}_n)$ exists and is a power of two.*

Using the metric

Let $\mathrm{Diam}(\mathsf{P})$ denote the maximum of the diameters of the atoms of P. A refining sequence will certainly separate if $\mathrm{Diam}(\mathsf{P}_n) \to 0$. So (2) will follow from

GOAL: *Given a partition R and $\varepsilon > 0$, there exists a partition $\mathsf{P} \succ \mathsf{R}$ with $\mathrm{Diam}(\mathsf{P}) \leqslant \varepsilon$ and $\mathrm{Deg}(\mathsf{P} \mid \mathsf{R})$ a power of two.*

3: Lemma.

a: *Suppose $E \subset X$ is non-void and clopen. Then for each posint K, there exists a K-set partition of E.*

b: *Given ε positive, there exists a partition P of X with $\mathrm{Diam}(P) \leqslant \varepsilon$.* ♢

Proof of (a). Here we use that X is perfect: Each non-void open subset is larger than a singleton.

To start the induction, set $E_1 := E$. Since E_{k-1} is non-void and clopen, it owns two distinct points. They can be separated by a clopen set and so there exist non-void clopen sets P_{k-1} and E_k such that

$$E_{k-1} = P_{k-1} \sqcup E_k.$$

Then $\{P_1, P_2, \ldots, P_{K-1}, E_K\}$ forms the desired partition. ♦

Proof of (b). Fix an $x \in X$ and let C be the complement of the open $[\varepsilon/2]$-ball about x. Each $y \in C$ is disconnected from x and so there is a clopen E_y owning y, with $x \in {E_y}^c$. Compactness of C gives a finite collection $F \subset C$ with $\bigcup_{y \in F} E_y \supset C$. Thus

$$D_x := \bigcap_{y \in F} {E_y}^c$$

is a clopen neighborhood of x with diameter less-equal ε.

By compactness, there exists a finite $F \subset X$ with $\bigcup_{z \in F} D_z = X$; we now have a finite cover of X by clopen sets of small diameter. For each $x \in X$, let P_x be the set formed by intersecting, over all $z \in F$, those D_z and those ${D_z}^c$ which own x. Then

$$\mathsf{P} := \{P_x \mid x \in X\}$$

is a partition with ε dominating its diameter. ♦

We can now prove the Theorem by establishing (GOAL). For each atom $E \in \mathsf{R}$, lemma (3b) yields a partition *of* E, call it P_E, such that $\mathrm{Diam}(\mathsf{P}_E) \leqslant \varepsilon$. Let k be a power-of-two which exceeds

$$\mathrm{Max}\{{}^{\#}(\mathsf{P}_E) \mid E \in \mathsf{R}\}$$

For each E we can apply (3a) and split one atom of P_E into further pieces to arrange that ${}^{\#}(\mathsf{P}_E)$ equal k; this can only decreases the diameter of a P_E.

Finally, let P denote the partition of X formed by the atoms of the $\{\mathsf{P}_E \mid E \in \mathsf{R}\}$.